Dominating cycles and forbidden pairs containing a path of order 5

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Abstract

A cycle C in a graph G is dominating if every edge of G is incident with a vertex of C. For a set $\mathcal{H}$ of connected graphs, a graph G is said to be $\mathcal{H}$-free if G does not contain any member of $\mathcal{H}$ as an induced subgraph. When $|\mathcal{H}| = 2$, $\mathcal{H}$ is called a forbidden pair. In this paper, we investigate the characterization of the class of the forbidden pairs guaranteeing the existence of a dominating cycle and show the following two results: (i) Every 2-connected $\{P_5, K_4^-\}$-free graph contains a longest cycle which is a dominating cycle. (ii) Every 2-connected $\{P_5, W^*\}$-free graph contains a longest cycle which is a dominating cycle. Here P_5 is the path of order 5, K_4^- is the graph obtained from the complete graph of order 4 by removing one edge, and W^* is a graph obtained from two triangles and an edge by identifying one vertex in each.

Keywords: Dominating cycles, Forbidden subgraphs, Forbidden pairs
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1 Introduction

In this paper, we consider only finite simple graphs. For terminology and notation not defined in this paper, we refer the readers to [4]. A graph G is said to be *Hamiltonian* if G has a *Hamilton*

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cycle, i.e., a cycle containing all vertices of G. A cycle C in a graph G is *dominating* if every edge of G is incident with a vertex of C.

Let $\mathcal{H}$ be a set of connected graphs. A graph G is said to be *$\mathcal{H}$-free* if G does not contain H as an induced subgraph for all H in $\mathcal{H}$, and we call each graph H of $\mathcal{H}$ a *forbidden subgraph*. We call $\mathcal{H}$ a *forbidden pair* if $|\mathcal{H}| = 2$. When we consider $\mathcal{H}$-free graphs, we assume that each member of $\mathcal{H}$ has order at least 3 because K_2 is the only connected graph of order 2 and K_1 is the unique K_2-free connected graph (here K_n denotes the complete graph of order n). In order to state results clearly, we further introduce the following notation. For two sets $\mathcal{H}_1$ and $\mathcal{H}_2$ of connected graphs, we write $\mathcal{H}_1 \leq \mathcal{H}_2$ if for every graph H_2 in $\mathcal{H}_2$, there exists a graph H_1 in $\mathcal{H}_1$ such that H_1 is an induced subgraph of H_2. Note that if $\mathcal{H}_1 \leq \mathcal{H}_2$, then every $\mathcal{H}_1$-free graph is also $\mathcal{H}_2$-free.

The forbidden pairs that force the existence of a Hamilton cycle in 2-connected graphs had been studied in [2, 5, 7]. In 1991, a characterization of such pairs was accomplished by Bedrossian [1]. Later, Faudree and Gould [6] extended the result of Bedrossian by regarding finite number of 2-connected $\{H_1, H_2\}$-free non-Hamiltonian graphs as exceptions. Here let P_n denote the path of order n, and the graphs $K_{1,3}$ (or claw), Z_n, $B_{m,n}$ and $N_{l,m,n}$ are the ones that are depicted in Figure 1.

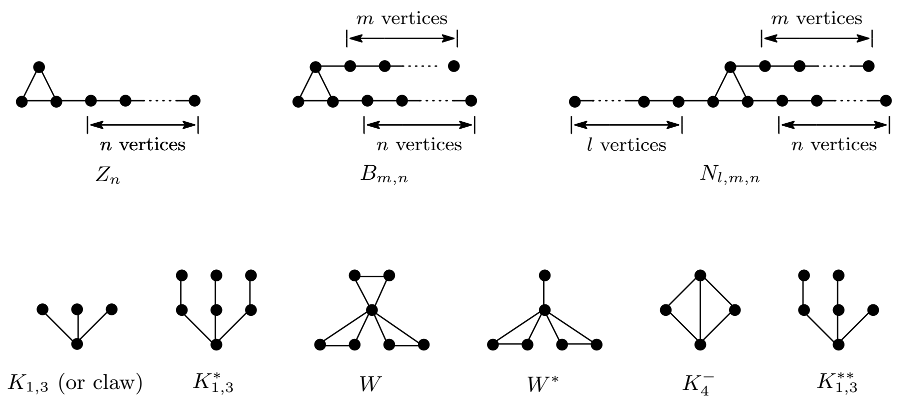


Figure 1: Forbidden subgraphs

Theorem A (Faudree and Gould [6]) *Let $\mathcal{H}$ be a forbidden pair. Then every 2-connected $\mathcal{H}$-free graph of sufficiently large order is Hamiltonian if and only if $\mathcal{H} \leq \{K_{1,3}, P_6\}$, $\mathcal{H} \leq \{K_{1,3}, Z_3\}$, $\mathcal{H} \leq \{K_{1,3}, B_{1,2}\}$, or $\mathcal{H} \leq \{K_{1,3}, N_{1,1,1}\}$.*

The purpose of this paper is to consider the analogue of Theorem A for dominating cycles which are relaxed structures of a Hamilton cycle. More precisely, we consider the following problem.

Problem 1 *Determine the set* $\boldsymbol{H}$ *(resp.,* $\boldsymbol{H}'$*) of forbidden pairs* $\mathcal{H}$ *which satisfy that every 2-connected* $\mathcal{H}$*-free graph (resp., every 2-connected* $\mathcal{H}$*-free graph of sufficiently large order) has a dominating cycle.*

Concerning this problem, the authors proved the following result in [3] (here let $K_{1,3}^*$, W, W^* and K_4^- be the ones that are depicted in Figure 1).

Theorem B ([3]) *Let* $\mathcal{H}$ *be a forbidden pair. If there exists a positive integer* $n_0 = n_0(\mathcal{H})$ *such that every 2-connected* $\mathcal{H}$*-free graph of order at least* n_0 *has a dominating cycle, then* $\mathcal{H} \leq \{K_{1,3}, Z_4\}$, $\mathcal{H} \leq \{K_{1,3}, B_{1,2}\}$, $\mathcal{H} \leq \{K_{1,3}, N_{1,1,1}\}$, $\mathcal{H} \leq \{P_4, W\}$, $\mathcal{H} \leq \{K_{1,3}^*, Z_1\}$, $\mathcal{H} \leq \{P_5, W^*\}$, *or* $\mathcal{H} \leq \{P_5, K_4^-\}$.

In the same paper, the authors also conjectured that the converse of Theorem B holds and gave a partial solution of the conjecture as follows. Here $K_{1,3}^{**}$ is the graph obtained from $K_{1,3}^*$ by deleting one leaf (see Figure 1).

Theorem C ([3]) *If* $\mathcal{H} \leq \{K_{1,3}, Z_4\}$, $\mathcal{H} \leq \{K_{1,3}, B_{1,2}\}$, $\mathcal{H} \leq \{K_{1,3}, N_{1,1,1}\}$, $\mathcal{H} \leq \{P_4, W\}$, *or* $\mathcal{H} \leq \{K_{1,3}^{**}, Z_1\}$, *then every 2-connected* $\mathcal{H}$*-free graph has a dominating cycle.*

In this paper, we show that the above conjecture is also true for the cases where $\mathcal{H} \leq \{P_5, W^*\}$ and $\mathcal{H} \leq \{P_5, K_4^-\}$ by considering slightly stronger statements.

Theorem 1 *Every 2-connected* $\{P_5, W^*\}$*-free graph contains a longest cycle which is a dominating cycle.*

Theorem 2 *Every 2-connected* $\{P_5, K_4^-\}$*-free graph contains a longest cycle which is a dominating cycle.*

Remark 1 *By Theorems B, C, 1 and 2, the remaining problem is only that whether the pair* $\{K_{1,3}^*, Z_1\}$ *belongs to the class* $\boldsymbol{H}$ *(resp.,* $\boldsymbol{H}'$*) of Problem 1 or not. Olariu [8] showed that if a connected* Z_1*-free graph* G *contains a triangle, then* G *is a complete multipartite graph. On the other hand, it is easy to check that every 2-connected complete multipartite graph containing a triangle has a dominating cycle. Thus the pair* $\{K_{1,3}^*, Z_1\}$ *belongs to the class* $\boldsymbol{H}$ *(resp.,* $\boldsymbol{H}'$*) if and only if the pair* $\{K_{1,3}^*, K_3\}$ *belongs to the class* $\boldsymbol{H}$ *(resp.,* $\boldsymbol{H}'$*). Consequently, we can deduce the target pair to* $\{K_{1,3}^*, K_3\}$*. Although we do not know the answer at the moment, we believe that the pair* $\{K_{1,3}^*, K_3\}$ *belongs to the class.*

In Section 2, we will introduce the lemmas in order to show Theorems 1 and 2, and we prove Theorems 1 and 2 in Sections 3 and 4, respectively.

2 Preparation for the proofs of Theorems 1 and 2

In this section, we prepare lemmas which will be used in the proofs of Theorems 1 and 2. To do that, we first prepare terminology and notation which we use in the rest.

Let G be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively, and let $|G| = |V(G)|$. For $X \subseteq V(G)$, we let $G[X]$ denote the subgraph induced by X in G, and let $G - X = G[V(G) \setminus X]$. Let v be a vertex of G. We denote by $N_G(v)$ the neighborhood of v in G. For $X \subseteq V(G) \setminus \{v\}$, we let $N_G(v; X) = N_G(v) \cap X$, and for $V, X \subseteq V(G)$ with $V \cap X = \emptyset$, let $N_G(V; X) = \bigcup_{v \in V} N_G(v; X)$. In this paper, we often identify a subgraph F of G with its vertex set $V(F)$ (for example, $N_G(v; V(F))$ is often denoted by $N_G(v; F)$).

A path with ends u and v is denoted by a (u, v)*-path*. For a subgraph H of G, a path P of G such that $|P| \geq 2$ is called a H*-path* if ends of P only belong to H. We write a cycle (or a path) C with a given orientation by $\overrightarrow{C}$. If there exists no chance of confusion, we abbreviate $\overrightarrow{C}$ by C. Let $\overrightarrow{C}$ be an oriented cycle or a path. For $u, v \in V(C)$, we denote by $u\overrightarrow{C}v$ the (u, v)-path on $\overrightarrow{C}$. The reverse sequence of $u\overrightarrow{C}v$ is denoted by $v\overleftarrow{C}u$. For $v \in V(C)$, we denote the h-th successor and the h-th predecessor of v on $\overrightarrow{C}$ by v^{+h} and v^{-h}, respectively, and let $v^{+0} = v^{-0} = v$. For $X \subseteq V(C)$, we define $X^{+h} = \{v^{+h} : v \in X\}$ and $X^{-h} = \{v^{-h} : v \in X\}$, respectively. We abbreviate v^{+1}, v^{-1}, X^{+1} and X^{-1} by v^+, v^-, X^+ and X^-, respectively.

2.1 Lemmas for P_5-free graphs

In this subsection, we give the following two lemmas (Lemmas 1 and 2) to make it easy to use the assumption "P_5-free" in the proofs of Theorems 1 and 2.

Lemma 1 *Let G be a graph, and let Q_1 and Q_2 be paths of order at least 3 with a common end a such that $Q_1 - a$ and $Q_2 - a$ are vertex-disjoint. If G is P_5-free and Q_1 is an induced path, then $N_G(Q_1 - a; Q_2 - a) \neq \emptyset$ or $V(Q_2) \setminus \{a\} \subseteq N_G(a)$.*

Proof of Lemma 1. Suppose that $N_G(Q_1 - a; Q_2 - a) = \emptyset$ and $V(Q_2) \setminus \{a\} \not\subseteq N_G(a)$. Write $Q_1 = a_1 a_2 \dots a_l$ and $Q_2 = a'_1 a'_2 \dots a'_{l'}$, where $a_1 = a'_1 = a$. Let i $(1 \leq i \leq l')$ be the minimum index with $aa'_i \notin E(G)$. Note that $a \neq a'_{i-1}$ and $aa'_{i-1} \in E(G)$. Hence $a_3 a_2 a a'_{i-1} a'_i$ is an induced path of G because Q_1 is an induced path and $N_G(Q_1 - a; Q_2 - a) = \emptyset$, which is a contradiction. $\square$

By Lemma 1, we can easily obtain the following.

Lemma 2 *Let G be a P_5-free graph, $\overrightarrow{C}$ be a cycle and H be a component of $G - C$, and let $v \in N_G(H; C)$ such that $V(H) \setminus N_G(v) \neq \emptyset$. If $N_G(H; v^+\overrightarrow{C}a) = \emptyset$ for some $a \in V(C) \setminus$*

$\{v, v^+\}$ *(resp.* $N_G(H; v^- \overleftarrow{C} a) = \emptyset$ *for some* $a \in V(C) \setminus \{v, v^-\}$*), then* $V(v^+ \overrightarrow{C} a) \subseteq N_G(v)$ *(resp.* $V(v^- \overleftarrow{C} a) \subseteq N_G(v)$*).*

Proof of Lemma 2. By the symmetry, it suffice to consider the case where $N_G(H; v^+ \overrightarrow{C} a) = \emptyset$ for some $a \in V(C) \setminus \{v, v^+\}$. Since $V(H) \setminus N_G(v) \neq \emptyset$, there exist two vertices $u, u' \in V(H)$ such that $vu, uu' \in E(G)$ and $vu' \notin E(G)$. Now we take two paths $Q_1 = vuu'$ and $Q_2 = v\overrightarrow{C} a$. Then Q_1 is an induced path of G and $N_G(Q_1 - v; Q_2 - v) = \emptyset$. This together with Lemma 1 leads to $V(v^+ \overrightarrow{C} a) = V(Q_2) \setminus \{v\} \subseteq N_G(v)$. $\square$

2.2 Properties of longest cycles in graphs

In this subsection, we introduce the basic lemmas concerning the properties of longest cycles in graphs.

We fix the following notation in this subsection. Let G be a graph and $\overrightarrow{C}$ be a longest cycle of G, and let H be a component of $G - C$. Then the following two lemmas hold (Lemmas 3 and 4). Since the proofs directly follow from the maximality of $|C|$, we omit it (see also Figure 2).

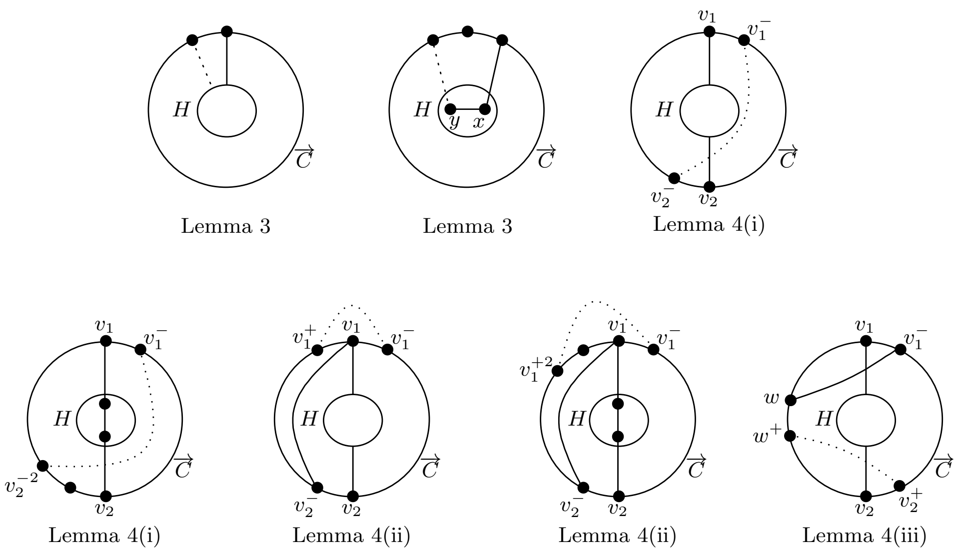


Figure 2: Longest cycles in graphs

Lemma 3 *$N_G(x; C) \cap N_G(y; C)^- = \emptyset$ for $x, y \in V(H)$. In particular, if $x \neq y$, then $N_G(x; C) \cap N_G(y; C)^{-2} = \emptyset$.*

Lemma 4 *Let v_1 and v_2 be two distinct vertices in $N_G(H; C)$. Then the following hold.*

(i) *There exists no C-path joining v_1^- and v_2^-, and joining v_1^+ and v_2^+, respectively; in particular, $E(G) \cap \{v_1^- v_2^-, v_1^+ v_2^+\} = \emptyset$. Moreover, if $|N_G(v_1; H) \cup N_G(v_2; H)| \geq 2$, then there exists no C-path joining v_1^- and v_2^{-2}, and joining v_1^+ and v_2^{+2}, respectively; in particular, $E(G) \cap \{v_1^- v_2^{-2}, v_1^+ v_2^{+2}\} = \emptyset$.*

(ii) *If $v_1 v_2^- \in E(G)$, then $v_1^- v_1^+ \notin E(G)$. Moreover, if $v_1 v_2^- \in E(G)$ and $|N_G(v_1; H) \cup N_G(v_2; H)| \geq 2$, then $E(G) \cap \{v_1^- v_1^{+2}, v_1^{-2} v_1^+\} = \emptyset$.*

(iii) *If $v_1^- w \in E(G)$ for some vertex w in $v_1 \overrightarrow{C} v_2^-$, then $v_2^+ w^+ \notin E(G)$. If $v_1^+ w \in E(G)$ for some vertex w in $v_1 \overleftarrow{C} v_2^+$, then $v_2^- w^- \notin E(G)$.*

2.3 Longest cycles in P_5-free graphs having no dominating longest cycle

For a cycle C of a graph G, let $\mu(C) = \max\{|F| : F$ is a component of $G - C\}$, and we define $\omega(C) = |\{F : F$ is a component of $G - C$ such that $|F| = \mu(C)\}|$.

Now let G be a graph, and we suppose that any longest cycles of G are not dominating cycles (i.e., $\mu(C) \geq 2$ for every longest cycle C of G), and let $\overrightarrow{C}$ be a longest cycle of G. Suppose further that C was chosen so that

(C1) $\mu(C)$ is as small as possible, and

(C2) $\omega(C)$ is as small as possible, subject to (C1).

Let H be a component of $G - C$ such that $|H| = \mu(C)$ (≥ 2).

Lemma 5 *If S is an independent set of G such that $S \subseteq V(C)$ and $N_G(S; G - C) = \emptyset$, then there exists no longest cycle D of G such that $V(D) \supseteq V(C) \setminus S$ and $V(D) \cap V(H) \neq \emptyset$.*

Proof of Lemma 5. Suppose that there exists a longest cycle D of G such that $V(D) \supseteq V(C) \setminus S$ and $V(D) \cap V(H) \neq \emptyset$. Let H' be an arbitrary component of $G - D$. By the assumptions of S, we see that $|H'| = 1$ or H' is an induced subgraph of some component of $G - C$ because $V(C) \setminus S \subseteq V(D)$. This implies that $\mu(D) \leq \mu(C)$. Moreover, if $\mu(D) = \mu(C)$, then the number of components of order $\mu(C)$ in $G - D$ is less than $\omega(C)$ because $V(D) \cap V(H) \neq \emptyset$. This contradicts the choice (C1) or (C2). $\Box$

By Lemma 5, the following two lemmas hold for P_5-free graphs.

Lemma 6 *If G is P_5-free, then $N_G(H; C) \cap N_G(H; C)^{-2} = \emptyset$.*

Proof of Lemma 6. Suppose that $N_G(H; C) \cap N_G(H; C)^{-2} \neq \emptyset$, and let $v \in N_G(H; C) \cap N_G(H; C)^{-2}$. Note that by Lemma 3, $N_G(v^+; H) = \emptyset$. Note also that G contains a longest cycle D such that $V(D) \supseteq V(C) \setminus \{v^+\}$ and $V(D) \cap V(H) \neq \emptyset$ because $v, v^{+2} \in N_G(H; C)$. Hence by Lemma 5, there exists a component H' of $G - C$ such that $H' \neq H$ and $N_G(v^+; H') \neq \emptyset$.

Let $x \in N_G(v; H)$, $y \in N_H(x)$ and $z \in N_G(v^+; H')$ (see Figure 3). Consider the paths $Q_1 = vxy$ and $Q_2 = vv^+z$. Since $v, v^{+2} \in N_G(H; C)$, it follows from Lemma 3 that $vy \notin E(G)$, and thus Q_1 is an induced path. Hence by Lemma 1, $N_G(\{x, y\}; \{v^+, z\}) = N_G(Q_1 - v; Q_2 - v) \neq \emptyset$ or $zv \in E(G)$. Since $N_G(v^+; H) = N_G(H; H') = \emptyset$, we have $zv \in E(G)$, but this contradicts Lemma 3. $\square$

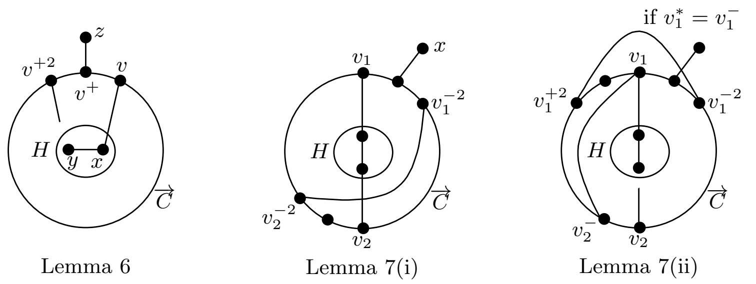


Figure 3: Lemmas 6 and 7

Lemma 7 *Let v_1 and v_2 be two distinct vertices in $N_G(H; C)$ such that $|N_G(v_1; H) \cup N_G(v_2; H)| \geq 2$, and suppose that G is P_5-free. Then the following hold.*

(i) $v_1^{-2} v_2^{-2} \notin E(G)$.

(ii) *If $V(H) \setminus N_G(v_1) \neq \emptyset$ and $v_1 v_2^- \in E(G)$, then $v_1^{-2} v_1^{+2} \notin E(G)$.*

Proof of Lemma 7. Note that v_1, v_1^-, v_1^{-2}, v_2, v_2^- and v_2^{-2} are distinct vertices by Lemma 3. Let $\overrightarrow{P}$ be a (v_1, v_2)-path such that $|P| \geq 4$ and $V(P) \setminus \{v_1, v_2\} \subseteq V(H)$.

To show (i), suppose that $v_1^{-2} v_2^{-2} \in E(G)$, and let $\overrightarrow{D} = v_1^{-2} v_2^{-2} \overleftarrow{C} v_1 \overrightarrow{P} v_2 \overrightarrow{C} v_1^{-2}$. Then D is a cycle in G such that $V(D) = (V(C) \setminus \{v_1^-, v_2^-\}) \cup V(P)$. Hence by the maximality of $|C|$, $|P| = 4$. Since $\{v_1^-, v_2^-\}$ is an independent set of G by Lemma 4(i) and since D is also a longest cycle of G, it follows from Lemma 5 that $N_G(v_i^-; G - C) \neq \emptyset$ for some i with $i \in \{1, 2\}$. Suppose that $N_G(v_1^-; G - C) \neq \emptyset$, and let $x \in N_G(v_1^-; G - C)$. Note that by Lemma 3, $x \notin V(H)$ (see Figure 3). Consider the paths $Q_1 = v_1^{-2} v_1^- x$ and $Q_2 = v_1^{-2} v_2^{-2} v_2^-$. By Lemma 3, Q_1 is an induced path. By Lemma 4(i), Q_2 is also an induced path. Hence by Lemma 1, $N_G(\{v_1^-, x\}; \{v_2^-, v_2^{-2}\}) \neq \emptyset$, but this contradicts Lemma 4(i). Thus $N_G(v_1^-; G - C) = \emptyset$. By the symmetry of v_1 and v_2, we can get a contradiction for the case where $N_G(v_2^-; G - C) \neq \emptyset$. Thus (i) holds.

To show (ii), suppose next that $V(H) \setminus N_G(v_1) \neq \emptyset$ and $\{v_1 v_2^-, v_1^{-2} v_1^{+2}\} \subseteq E(G)$, and let $\overrightarrow{D'} = v_2^- v_1 \overrightarrow{P} v_2 \overrightarrow{C}\ v_1^{-2} v_1^{+2} \overrightarrow{C} v_2^-$. Then D' is a cycle in G such that $V(D) = (V(C) \setminus \{v_1^-, v_2^-\}) \cup V(P)$, and the maximality of $|C|$ implies that $|P| = 4$. Since $\{v_1^-, v_1^+\}$ is an independent set of G by Lemma 4(ii) and since D' is a longest cycle of G, it follows from Lemmas 3 and 5 that

$N_G(v_1^*; H') \neq \emptyset$ for some $v_1^* \in \{v_1^-, v_1^+\}$ and some component H' of $G - C$ with $H' \neq H$ (see Figure 3). Since $V(H) \setminus N_G(v_1) \neq \emptyset$, $G[V(H) \cup \{v_1\}]$ contains an induced path Q_1' of order at least 3 with an end v_1. By Lemma 3, $G[V(H') \cup \{v_1, v_1^*\}]$ contains an induced path Q_2' of order at least 3 with an end v_1 and $v_1 v_1^* \in E(Q_2')$. Hence by Lemma 1 and since $N_G(H; H') = \emptyset$, we see that $N_G(v_1^*; H) \neq \emptyset$, which contradicts Lemma 3. Thus (ii) also holds. $\square$

3 Proof of Theorem 1

Let G be a 2-connected $\{P_5, W^*\}$-free graph, and we show that G contains a longest cycle which is a dominating cycle. By way of a contradiction, suppose that any longest cycles of G are not dominating cycles. Let $\overrightarrow{C}$ be the same described as in the paragraph preceding Lemma 5 in Subsection 2.3, and let H be a component of $G - C$ such that $|H| = \mu(C)$ ($\geq$ 2). Since G is 2-connected, there exist two distinct vertices v_1 and v_2 in $N_G(H; C)$ such that $|N_G(v_1; H) \cup N_G(v_2; H)| \geq 2$. Then, $|v_i \overrightarrow{C} v_{3-i}| \geq 4$ for $i \in \{1, 2\}$ because C is longest. (Note that by these assumptions, in this proof, we can use all lemmas of Section 2.) We choose the vertices v_1 and v_2 so that

$$N_G(H; v_1^+ \overrightarrow{C} v_2^-) = \emptyset. \tag{3.1}$$

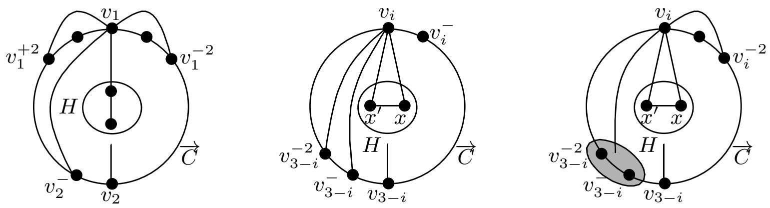
Claim 3.1 $V(H) \subseteq N_G(v_1) \cap N_G(v_2)$.

Proof. Suppose that $V(H) \setminus N_G(v_1) \neq \emptyset$. Then by Lemma 2 and (3.1), we have $\{v_1 v_1^{+2}, v_1 v_2^-\} \subseteq E(G)$. Moreover, by Lemmas 3 and 6, $N_G(H; \{v_1^-, v_1^{-2}\}) = \emptyset$, and hence Lemma 2 yields that $v_1 v_1^{-2} \in E(G)$. Since $v_1 v_2^- \in E(G)$, it follows from Lemma 4(ii) that $E(G) \cap \{v_1^- v_1^+, v_1^{-2} v_1^+, v_1^- v_1^{+2}\} = \emptyset$. By Lemma 7(ii), we also have $v_1^{-2} v_1^{+2} \notin E(G)$. Therefore, since $N_G(H; \{v_1^-, v_1^{-2}, v_1^+, v_1^{+2}\}) = \emptyset$ by Lemmas 3 and 6, we see that $G[\{v_1, v_1^-, v_1^{-2}, v_1^+, v_1^{+2}\} \cup N_G(v_1; H)]$ contains a W^* as an induced subgraph (see Figure 4), a contradiction. Thus $V(H) \subseteq N_G(v_1)$. Similarly, we have $V(H) \subseteq N_G(v_2)$. $\square$

Claim 3.2 *For $i \in \{1, 2\}$, $|E(G) \cap \{v_i v_{3-i}^-, v_i v_{3-i}^{-2}\}| \leq 1$.*

Proof. Suppose that $\{v_i v_{3-i}^-, v_i v_{3-i}^{-2}\} \subseteq E(G)$, and let $xx' \in E(H)$. By Claim 3.1, $\{xv_i, x'v_i\} \subseteq E(G)$ (see Figure 4). By Lemmas 3 and 6, we have $E(G) \cap \{xv_i^-, xv_{3-i}^-, xv_{3-i}^{-2}, x'v_i^-, x'v_{3-i}^-, x'v_{3-i}^{-2}\} = \emptyset$. By Lemma 4(i), we also have $E(G) \cap \{v_i^- v_{3-i}^-, v_i^- v_{3-i}^{-2}\} = \emptyset$. This implies that $G[\{v_i, v_i^-, v_{3-i}^-, v_{3-i}^{-2}, x, x'\}] \cong W^*$, a contradiction. Thus $|E(G) \cap \{v_i v_{3-i}^-, v_i v_{3-i}^{-2}\}| \leq 1$. $\square$

Claim 3.3 *For $i \in \{1, 2\}$, if $E(G) \cap \{v_i v_{3-i}^-, v_i v_{3-i}^{-2}\} \neq \emptyset$, then $v_i v_i^{-2} \notin E(G)$.*



Claim 3.1 Claim 3.2 Claim 3.3

Figure 4: Claims 3.1–3.3

Proof. Let $v^* \in \{v_{3-i}^-, v_{3-i}^{-2}\}$, and we show that if $v_iv^* \in E(G)$, then $v_iv_i^{-2} \notin E(G)$. By way of a contradiction, suppose that $\{v_iv^*, v_iv_i^{-2}\} \subseteq E(G)$, and let $xx' \in E(H)$. By Claim 3.1, $\{xv_i, x'v_i\} \subseteq E(G)$ (see Figure 4). By Lemmas 4(i) and 7(i), $E(G) \cap \{v_i^-v^*, v_i^{-2}v^*\} = \emptyset$. By Lemmas 3 and 6, $E(G) \cap \{xv_i^-, xv_i^{-2}, xv^*, x'v_i^-, x'v_i^{-2}, x'v^*\} = \emptyset$. This implies that $G[\{v_i, v_i^-, v_i^{-2}, x, x', v^*\}] \cong W^*$, a contradiction. Thus if $v_iv^* \in E(G)$, then $v_iv_i^{-2} \notin E(G)$. $\square$

Claim 3.4 $v_1v_2 \in E(G)$.

Proof. Let $x \in V(H)$, and consider the paths $Q_1 = xv_1v_1^-$ and $Q_2 = xv_2v_2^-$. By Lemma 3, each Q_i is an induced path. Hence by Lemma 1, $N_G(\{v_1, v_1^-\}; \{v_2, v_2^-\}) \neq \emptyset$. Since $v_1^-v_2^- \notin E(G)$ by Lemma 4(i), we have $E(G) \cap \{v_1v_2^-, v_1^-v_2, v_1v_2\} \neq \emptyset$.

Suppose that $v_iv_{3-i}^- \in E(G)$ for some $i \in \{1, 2\}$, and next consider the paths $Q'_1 = v_iv_i^-v_i^{-2}$ and $Q'_2 = v_iv_{3-i}^-v_{3-i}^{-2}$. It follows from Claims 3.2 and 3.3 that each Q'_i is an induced path. Hence by Lemma 1, $N_G(\{v_i^-, v_i^{-2}\}; \{v_{3-i}^-, v_{3-i}^{-2}\}) \neq \emptyset$, which contradicts Lemma 4(i) or Lemma 7(i). Thus $v_1v_2^- \notin E(G)$ for $i \in \{1, 2\}$, and hence $v_1v_2 \in E(G)$. $\square$

Claim 3.5 *For* $i \in \{1, 2\}$, $v_iv_i^{-2} \in E(G)$.

Proof. Suppose that $v_iv_i^{-2} \notin E(G)$, and consider the paths $Q_1 = v_iv_i^-v_i^{-2}$ and $Q_2 = v_iv_{3-i}v_{3-i}^-$ (note that by Claim 3.4, $v_1v_2 \in E(G)$). Then Q_1 is an induced path. Hence by Lemma 1, $N_G(\{v_i^-, v_i^{-2}\}; \{v_{3-i}, v_{3-i}^-\}) \neq \emptyset$ or $v_iv_{3-i}^- \in E(G)$. This together with Lemma 4(i) implies that $E(G) \cap \{v_iv_{3-i}^-, v_i^-v_{3-i}, v_i^{-2}v_{3-i}\} \neq \emptyset$.

Assume first that $v_i^-v_{3-i} \in E(G)$ or $v_i^{-2}v_{3-i} \in E(G)$. Then by Claim 3.2, $G[\{v_{3-i}, v_i^-, v_i^{-2}\}]$ contains an induced path Q'_1 of order 3 with an end v_{3-i}. By Claim 3.3, we also see that $Q'_2 = v_{3-i}v_{3-i}^-v_{3-i}^{-2}$ is an induced path. Hence by Lemma 1, $N_G(\{v_i^-, v_i^{-2}\}; \{v_{3-i}^-, v_{3-i}^{-2}\}) \neq \emptyset$, which contradicts Lemma 4(i) or Lemma 7(i). Thus $E(G) \cap \{v_i^-v_{3-i}, v_i^{-2}v_{3-i}\} = \emptyset$, and hence $v_iv_{3-i}^- \in E(G)$. We now consider the paths Q_1 and $Q''_2 = v_iv_{3-i}^-v_{3-i}^{-2}$. Then by Claim 3.2, Q''_2 is an induced path. Hence by Lemma 1, $N_G(\{v_i^-, v_i^{-2}\}; \{v_{3-i}^-, v_{3-i}^{-2}\}) \neq \emptyset$, which contradicts

Lemma 4(i) or Lemma 7(i) again. Thus $v_i v_i^{-2} \in E(G)$. $\square$

Now we choose a longest cycle $\overrightarrow{C}$, a component H and vertices v_1 and v_2 such that $|N_G(v_1; H) \cup N_G(v_2; H)| \geq 2$ and $N_G(H; v_1^+ \overrightarrow{C} v_2^-) = \emptyset$ so that

(C3) $|v_1 \overrightarrow{C} v_2|$ is as large as possible, subject to (C1) and (C2).

Then by the choice, we can easily obtain the following.

Claim 3.6 $v_1^- v_1^+ \notin E(G)$.

Proof. Note that by Lemma 3, $N_G(H; \{v_1^-, v_1^+\}) = \emptyset$. If $v_1^- v_1^+ \in E(G)$, then since $v_1 v_1^{-2} \in E(G)$ by Claim 3.5, $D = v_1^{-2} v_1 v_1^- v_1^+ \overrightarrow{C} v_1^{-2}$ is a cycle in G such that $V(D) = V(C)$, $N_G(H; v_1^+ \overrightarrow{D} v_2^-) = \emptyset$ and $|v_1 \overrightarrow{D} v_2| > |v_1 \overrightarrow{C} v_2|$, which contradicts the choice (C3). $\square$

Claim 3.7 $v_1^{-2} v_1^+ \in E(G)$.

Proof. Let $xx' \in E(H)$. Note that by Claims 3.1 and 3.5, $\{xv_1, x'v_1, v_1 v_1^{-2}\} \subseteq E(G)$. By Lemmas 3 and 6, $E(G) \cap \{xv_1^{-2}, xv_1^-, xv_1^+, x'v_1^{-2}, x'v_1^-, x'v_1^+\} = \emptyset$. By Claim 3.6, we also have $v_1^- v_1^+ \notin E(G)$. Therefore, if $v_1^{-2} v_1^+ \notin E(G)$, then $G[\{v_1, v_1^-, v_1^{-2}, x, x', v_1^+\}] \cong W^*$, a contradiction. $\square$

Note that by Lemma 4(i) and Claim 3.7, $|v_2 \overrightarrow{C} v_1| \geq 6$. By Lemma 4(i), we have

$$E(G) \cap \{v_1^- v_2^-, v_1^{-2} v_2^-\} = \emptyset. \tag{3.2}$$

Since $v_1^{-2} v_1^+ \in E(G)$ by Claim 3.7, it follows from Lemma 4(iii) that

$$v_1^{-3} v_2^- \notin E(G). \tag{3.3}$$

Since $v_i v_i^{-2} \in E(G)$ for $i \in \{1, 2\}$ by Claim 3.5, it follows from Claim 3.3 that

$$E(G) \cap \{v_1 v_2^-, v_2 v_1^-, v_2 v_1^{-2}\} = \emptyset. \tag{3.4}$$

Consider the paths $Q_1 = v_1 v_2 v_2^-$ and $Q_2 = v_1 v_1^{-2} v_1^{-3}$. By (3.4), Q_1 is an induced path. Hence by Lemma 1, $N_G(\{v_2, v_2^-\}; \{v_1^{-2}, v_1^{-3}\}) \neq \emptyset$ or $v_1 v_1^{-3} \in E(G)$. This together with (3.2)–(3.4) implies that $E(G) \cap \{v_1 v_1^{-3}, v_2 v_1^{-3}\} \neq \emptyset$. Note that by Lemmas 3 and 6, $N_G(H; \{v_1^{-2}, v_1^-\}) = \emptyset$. Note also that by Claim 3.7, $v_1^{-2} v_1^+ \in E(G)$. Therefore, if $v_1^{-3} v_1 \in E(G)$, then $D = v_1^{-3} v_1 v_1^- v_1^{-2} v_1^+ \overrightarrow{C} v_1^{-3}$ is a cycle in G such that $V(D) = V(C)$, $N_G(H; v_1^+ \overrightarrow{D} v_2^-) = \emptyset$ and $|v_1 \overrightarrow{D} v_2| > |v_1 \overrightarrow{C} v_2|$, which contradicts the choice (C3). Thus $v_1^{-3} v_1 \notin E(G)$, and hence $v_1^{-3} v_2 \in E(G)$.

We next consider the paths $Q_1' = v_1^{-3} v_2 v_2^-$ and $Q_2' = v_1^{-3} v_1^{-2} v_1^-$. By (3.3), Q_1' is an induced path. Since $N_G(\{v_2, v_2^-\}; \{v_1^-, v_1^{-2}\}) = \emptyset$ by (3.2) and (3.4), it follows from Lemma 1 that

$v_1^- v_1^{-3} \in E(G)$. Note that by Claims 3.5 and 3.7, $\{v_1 v_1^{-2}, v_1^{-2} v_1^+\} \subseteq E(G)$, and hence $D = v_1^{-3} v_1^- v_1 v_1^{-2} v_1^+ \overrightarrow{C} v_1^{-3}$ is a cycle in G such that $V(D) = V(C)$, $N_G(H; v_1^+ \overrightarrow{D} v_2^-) = \emptyset$ and $|v_1 \overrightarrow{D} v_2| > |v_1 \overrightarrow{C} v_2|$, which contradicts the choice (C3).

This completes the proof of Theorem 1. $\square$

4 Proof of Theorem 2

Let G be a 2-connected $\{P_5, K_4^-\}$-free graph. We first introduce a useful claim for our proof.

Claim 4.1 *Let $\overrightarrow{Q}$ be a path of G starting from $v \in V(G)$ such that v is adjacent to every vertex in $V(Q) \setminus \{v\}$, and let $a \in V(G) \setminus V(Q)$. Then either $V(Q) \subseteq N_G(a)$ or $|N_G(a; Q)| \leq 1$.*

Proof. If $|Q| \leq 2$, then the assertion clearly holds. Thus we may assume that $|Q| \geq 3$.

We first suppose that $G[V(Q)]$ is not complete. Then there exist h, l with $1 \leq h < l \leq |Q|-1$ such that $v^{+h} v^{+l} \notin E(G)$. Choose h and l so that $l-h$ is as small as possible. Note that $l \geq h+2$ and $v^{+h} v^{+(h+1)}, v^{+(h+1)} v^{+l} \in E(G)$. Hence $\{v, v^{+h}, v^{+(h+1)}, v^{+l}\}$ induces K_4^- in G, which is a contradiction. Thus $G[V(Q)]$ is complete.

If $V(Q) \not\subseteq N_G(a)$ and $|N_G(a; Q)| \geq 2$, then there exist three vertices u, u' and u'' such that $u, u' \in N_G(a; Q)$ and $u'' \notin N_G(a; Q)$, and hence $\{a, u, u', u''\}$ induces K_4^- in G because $G[V(Q)]$ is complete, which is a contradiction. Consequently, we get the desired conclusion. $\square$

We show that G contains a longest cycle which is a dominating cycle. By way of a contradiction, suppose that any longest cycles of G are not dominating cycles. Let $\overrightarrow{C}$ be the same described as in the paragraph preceding Lemma 5 in Subsection 2.3, and let H be a component of $G - C$ such that $|H| = \mu(C)$ (≥ 2). Since G is 2-connected, there exist two distinct vertices v_1 and v_2 in $N_G(H; C)$ such that $|N_G(v_1; H) \cup N_G(v_2; H)| \geq 2$. Then $|v_i \overrightarrow{C} v_{3-i}| \geq 4$ for $i \in \{1, 2\}$ because C is longest. (Note that by these assumptions, we can use all lemmas of Section 2.) We choose the vertices v_1 and v_2 so that

$$N_G(H; v_1^+ \overrightarrow{C} v_2^-) = \emptyset. \tag{4.1}$$

Claim 4.2 *There exists an edge $x_1 x_2$ in H such that $v_i x_i \in E(G)$ for $i \in \{1, 2\}$.*

Proof. Suppose not. Let $x_1 \in N_G(v_1; H)$ and $x_2 \in N_G(v_2; H)$ be distinct vertices, and let P be a shortest (x_1, x_2)-path in H. We choose x_1 and x_2 so that $|P|$ is as small as possible. Then $x_1 x_2 \notin E(G)$ and $\emptyset \neq V(P) \setminus \{x_1, x_2\} \subseteq V(P) \setminus N_G(v_i)$ for $i \in \{1, 2\}$. Hence by Lemma 2 and (4.1), $V(v_1^+ \overrightarrow{C} v_2^-) \subseteq N_G(v_1) \cap N_G(v_2)$, and this implies that $v_1 v_2 \in E(G)$ (otherwise, $G[V(v_1 \overrightarrow{C} v_2)]$ contains a K_4^- as an induced subgraph, a contradiction). On the other hand, consider the paths $Q_1 = P$ and $Q_2 = x_1 v_1 v_1^+$. Then by the minimality of $|P|$, Q_1 is an induced

path of order at least 3. By Lemma 3, Q_2 is also an induced path. Hence by Lemma 1, $N_G(P - x_1; \{v_1, v_1^+\}) \neq \emptyset$. Combining this with (4.1) and the fact that $V(P) \setminus \{x_1, x_2\} \subseteq V(P) \setminus N_G(v_1)$, we get $v_1x_2 \in E(G)$. Similarly, by considering the paths P and $x_2v_2v_2^-$, we also have $v_2x_1 \in E(G)$. But then $G[\{v_1, v_2, x_1, x_2\}] \cong K_4^-$, a contradiction. $\square$

Let x_1x_2 be as in Claim 4.2. By the symmetry of $\overrightarrow{C}$ and $\overleftarrow{C}$, we may always assume that $v_1x_2 \notin E(G)$ if $\{v_1x_2, v_2x_1\} \not\subseteq E(G)$.

Now let w_1 be a vertex in $v_2^+\overrightarrow{C}v_1^-$ such that $V(w_1\overrightarrow{C}v_1^-) \subseteq N_G(v_1)$. We choose w_1 so that $|w_1\overrightarrow{C}v_1|$ is as large as possible. By Lemma 4(i), Claim 4.1 and the choice of w_1, we can easily obtain the following.

Claim 4.3 (i) *$|N_G(x; w_1\overrightarrow{C}v_1)| \leq 1$ for $x \in \{v_2^-, v_2^{-2}, x_1, x_2\}$.*

(ii) *If $w_1^- \neq v_2$, then $N_G(w_1^-; w_1\overrightarrow{C}v_1) = \{w_1\}$.*

(iii) *If $\{v_1v_2^-, v_2x_1\} \cap E(G) \neq \emptyset$, then $|N_G(v_2; w_1\overrightarrow{C}v_1)| \leq 1$.*

Proof. Let $x \in \{v_2^-, v_2^{-2}, x_1, x_2\}$. Then by Lemmas 3 and 4(i), $xv_1^- \notin E(G)$. Hence by applying Claim 4.1 as $\overrightarrow{Q} = v_1\overleftarrow{C}w_1$ and $a = x$, we have $|N_G(x; w_1\overrightarrow{C}v_1)| \leq 1$. Thus (i) holds. If $w_1^- \neq v_2$, then by the choice of w_1, $w_1^-v_1 \notin E(G)$, and hence again by Claim 4.1, $N_G(w_1^-; w_1\overrightarrow{C}v_1) = \{w_1\}$. Thus (ii) also holds. To show (iii), suppose that $\{v_1v_2^-, v_2x_1\} \cap E(G) \neq \emptyset$ and $|N_G(v_2; w_1\overrightarrow{C}v_1)| \geq 2$. Since $|N_G(v_2; w_1\overrightarrow{C}v_1)| \geq 2$, it follows from Claim 4.1 that $V(w_1\overrightarrow{C}v_1) \subseteq N_G(v_2)$, and thus $v_2w_1\overrightarrow{C}v_1$ is a path with an end v_2 such that $V(w_1\overrightarrow{C}v_1) \subseteq N_G(v_2)$ (see Figure 5). Since $\{v_1x_1, v_2v_2^-\} \subseteq E(G)$, the assumption $\{v_1v_2^-, v_2x_1\} \cap E(G) \neq \emptyset$ implies that $\{v_1, v_2\} \subseteq N_G(x)$ for some $x \in \{x_1, v_2^-\}$, and thus $|N_G(x; v_2w_1\overrightarrow{C}v_1)| \geq |\{v_1, v_2\}| = 2$. Then again by Claim 4.1, $V(v_2w_1\overrightarrow{C}v_1) \subseteq N_G(x)$, in particular, $xv_1^- \in E(G)$, which contradicts Lemma 3 or Lemma 4(i). Thus (iii) holds. $\square$

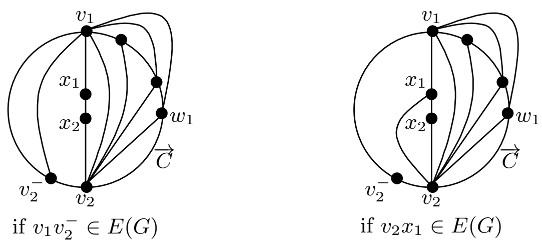


Figure 5: Claim 4.3(iii)

Since $v_1x_1 \in E(G)$, the following fact is directly obtained from Claim 4.3(i).

Fact 4.4 $N_G(x_1; w_1\overrightarrow{C}v_1^-) = \emptyset$.

We divide the proof into two cases according as $\{v_1x_2, v_2x_1\} \not\subseteq E(G)$ or $\{v_1x_2, v_2x_1\} \subseteq E(G)$.

Case 1. $\{v_1x_2, v_2x_1\} \not\subseteq E(G)$.

Then $v_1x_2 \notin E(G)$ (see the paragraph following the proof of Claim 4.2). Hence by Lemma 2 and (4.1),

$$\{v_2^-, v_2^{-2}\} \subseteq N_G(v_1). \tag{4.2}$$

Then by applying Claim 4.3(i) as $x = v_2^-$ and $x = v_2^{-2}$, the following fact holds.

Fact 4.5 *$N_G(v_2^{-h}; w_1\overrightarrow{C}v_1^-) = \emptyset$ for $h \in \{1, 2\}$.*

Moreover, by Lemmas 3 and 6, $N_G(H; \{v_1^-, v_1^{-2}\}) = \emptyset$, and hence Lemma 2 yields that $v_1v_1^{-2} \in E(G)$. This together with the choice of w_1 implies that

$$w_1^+ \neq v_1. \tag{4.3}$$

Claim 4.6 *$N_G(v_2; w_1\overrightarrow{C}v_1^-) = \emptyset$. In particular, $w_1^- \neq v_2$.*

Proof. Suppose that $N_G(v_2; w_1\overrightarrow{C}v_1^-) \neq \emptyset$. Then by Claim 4.3(iii) and (4.2), $|N_G(v_2; w_1\overrightarrow{C}v_1)| = |N_G(v_2; w_1\overrightarrow{C}v_1^-)| = 1$. By (4.3), the equality $|N_G(v_2; w_1\overrightarrow{C}v_1^-)| = 1$ implies that $G[V(w_1\overrightarrow{C}v_1^-) \cup \{v_2\}]$ contains an induced path Q_1 of order at least 3 with an end v_2. On the other hand, the equality $|N_G(v_2; w_1\overrightarrow{C}v_1)| = |N_G(v_2; w_1\overrightarrow{C}v_1^-)|$ implies that $v_1v_2 \notin E(G)$. Since $v_2v_2^- \in E(G)$ and $G[\{v_1, v_2^-, v_2^{-2}\}]$ is triangle by (4.2), these together with Claim 4.1 imply that $v_2v_2^{-2} \notin E(G)$ (see the left of Figure 6), and thus $Q_2 = v_2v_2^-v_2^{-2}$ is also an induced path. Hence by Lemma 1, $N_G(Q_1 - v_2; Q_2 - v_2) \neq \emptyset$. Since $N_G(Q_1 - v_2; Q_2 - v_2) \subseteq N_G(\{v_2^-, v_2^{-2}\}; w_1\overrightarrow{C}v_1^-)$, this contradicts Fact 4.5. $\square$

Claim 4.7 *$N_G(x_2; w_1\overrightarrow{C}v_1^-) = \emptyset$.*

Proof. Suppose not. Then by Claim 4.3(i), $|N_G(x_2; w_1\overrightarrow{C}v_1^-)| = 1$, and this together with (4.3) implies that $G[V(w_1\overrightarrow{C}v_1^-) \cup \{x_2\}]$ contains an induced path Q_1 of order at least 3 with an end x_2. On the other hand, by (4.1), the path $Q_2 = x_2v_2v_2^-$ is also an induced path. Therefore, by Lemma 1, we have $N_G(\{v_2, v_2^-\}; w_1\overrightarrow{C}v_1^-) \supseteq N_G(Q_1 - x_2; Q_2 - x_2) \neq \emptyset$, which contradicts Fact 4.5 or Claim 4.6. $\square$

Claim 4.8 *$\{x_1w_1^-, x_2w_1^-\} \subseteq E(G)$.*

Proof. Consider the paths $Q_1 = v_1w_1w_1^-$ and $Q_2 = v_1x_1x_2$. Since $w_1^- \neq v_2$ by Claim 4.6, it follows from Claim 4.3(ii) that Q_1 is an induced path. Since $v_1x_2 \notin E(G)$, Q_2 is also an induced path. Hence by Lemma 1, Fact 4.4 and Claim 4.7, it follows that $N_G(w_1^-; \{x_1, x_2\}) \neq \emptyset$. We

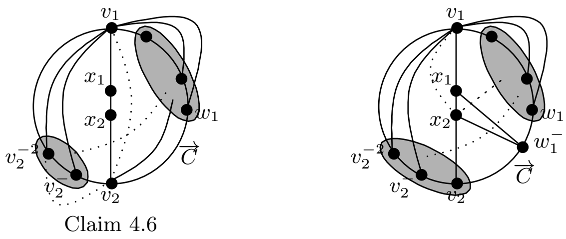


Figure 6: The cycle C in Case 1

next consider the path $w_1^+ w_1 w_1^-$ and a path in $G[\{w_1^-, x_1, x_2\}]$ of order 3 with an end w_1^-. By Claim 4.3(ii), the path $w_1^+ w_1 w_1^-$ is an induced path. Hence by again Lemma 1, Fact 4.4 and Claim 4.7, we can easily obtain $\{x_1 w_1^-, x_2 w_2^-\} \subseteq E(G)$. $\square$

The graph illustrated in the right of Figure 6 is a current situation. We now consider the paths $Q_1 = w_1^- w_1 w_1^+$ and $Q_2 = w_1^- x_2 v_2$. By Claims 4.3(ii) and 4.6, Q_1 is an induced path. Hence by Lemma 1 and Claims 4.6 and 4.7, we have $w_1^- v_2 \in E(G)$, and hence $|N_G(v_2; \{x_1, x_2, w_1^-\})| \geq |\{x_2, w_1^-\}| = 2$. Since $G[\{x_1, x_2, w_1^-\}]$ is a triangle by Claim 4.8, it follows from Claim 4.1 that $v_2 x_1 \in E(G)$. Therefore, we see that $G[\{v_2, x_1, x_2\}]$ is also a triangle. Since $\{v_2, x_1, x_2\} \not\subseteq N_G(v_1)$ because $v_1 x_2 \notin E(G)$, Claim 4.1 also yields that $v_1 v_2$ is not an edge in G.

On the other hand, consider the paths $Q_1' = v_2^- v_2 x_2$ and $Q_2' = v_2^- v_1 w_1$. Since $v_2^- x_2 \notin E(G)$ by (4.1), it follows that Q_1' is an induced path. By Fact 4.5, Q_2' is also an induced path. Hence by Lemma 1, $N_G(\{v_2, x_2\}; \{v_1, w_1\}) \neq \emptyset$. This together with Claims 4.6, 4.7 and the assumption $v_1 x_2 \notin E(G)$ implies that $v_1 v_2$ is an edge in G. This is a contradiction.

Case 2. $\{v_1 x_2, v_2 x_1\} \subseteq E(G)$.

By the assumption of Case 2, Claim 4.3(ii) and (iii), the following claim holds.

Claim 4.9 *$N_G(v_2; w_1 \overrightarrow{C} v_1) = \{v_1\}$ and $N_G(w_1^-; w_1 \overrightarrow{C} v_1) = \{w_1\}$.*

Proof. If $v_1 v_2 \notin E(G)$, then $G[\{v_1, v_2, x_1, x_2\}]$ is isomorphic to K_4^-, a contradiction. Thus $v_1 v_2 \in E(G)$. This together with Claim 4.3(iii) forces $N_G(v_2; w_1 \overrightarrow{C} v_1) = \{v_1\}$. In particular, $w_1^- \neq v_2$. Hence by Claim 4.3(ii), we have $N_G(w_1^-; w_1 \overrightarrow{C} v_1) = \{w_1\}$. $\square$

By Claim 4.9, $w_1^- v_1 \notin E(G)$. Hence by applying Claim 4.1 as $Q = v_1 x_1 x_2$ and $a = w_1^-$, it follows that $|N_G(w_1^-; \{v_1, x_1, x_2\})| \leq 1$. Therefore, by changing the label of x_1 and x_2 if necessary, we may assume that

$$w_1^- x_1 \notin E(G). \tag{4.4}$$

Let z_1 be a vertex in $v_1^+\overrightarrow{C}v_2^-$ such that $v_1z_1 \in E(G)$. Then

$$z_1 \neq v_2^- \tag{4.5}$$

(otherwise, by (4.1), Claim 4.9 and the assumption of Case 2, $G[\{v_1, v_2, x_1, z_1\}]$ is isomorphic to K_4^-, a contradiction). We choose z_1 so that $|v_1\overrightarrow{C}z_1|$ is as large as possible.

Claim 4.10 (i) $V(z_1^+\overrightarrow{C}v_2^-) \subseteq N_G(z_1)$.

(ii) $N_G(v_2; z_1\overrightarrow{C}v_2^-) = \{v_2^-\}$.

Proof. (i) If $z_1^+ = v_2^-$, then the assertion clearly holds. Thus we may assume that $z_1^+ \neq v_2^-$. Consider the paths $Q_1 = z_1v_1x_1$ and $Q_2 = z_1\overrightarrow{C}v_2^-$. Then by (4.1), Q_1 is an induced path. By the choice of z_1 and again (4.1), $N_G(Q_1 - z_1; Q_2 - z_1) = \emptyset$. Hence Lemma 1 yields that $V(z_1^+\overrightarrow{C}v_2^-) = V(Q_2 - z_1) \subseteq N_G(z_1)$.

(ii) By Claim 4.9, (4.5) and the choice of z_1, we see that $|z_1\overrightarrow{C}v_2| - 1 \geq |N_G(v_1; v_2\overleftarrow{C}z_1)| \geq |\{v_2, z_1\}| = 2$. Since $v_2\overleftarrow{C}z_1$ is a path with an end v_2, $V(v_2\overleftarrow{C}z_1) \not\subseteq N_G(v_1)$ and $|N_G(v_1; v_2\overleftarrow{C}z_1)| \geq 2$, it follows from Claim 4.1 that $V(z_1\overrightarrow{C}v_2^-) \not\subseteq N_G(v_2)$, i.e., $|N_G(v_2; z_1\overrightarrow{C}v_2^-)| \leq |z_1\overrightarrow{C}v_2^-| - 1$. By Claim 4.10(i), $z_1\overrightarrow{C}v_2^-$ is a path with an end z_1 such that $V(z_1^+\overrightarrow{C}v_2^-) \subseteq N_G(z_1)$, and hence Claim 4.1 implies that $N_G(v_2; z_1\overrightarrow{C}v_2^-) = \{v_2^-\}$. □

Claim 4.11 $N_G(v_2^{-h}; w_1^-\overrightarrow{C}v_1^-) = \emptyset$ *for* $h \in \{1, 2\}$.

Proof. Suppose that $N_G(v_2^{-h}; w_1^-\overrightarrow{C}v_1^-) \neq \emptyset$ for some $h \in \{1, 2\}$, and choose v_2^{-h} so that $h = 1$ if possible. Note that by Lemmas 4(i) and 7(i), $w_1 \neq v_1^-$.

We first assume that $N_G(v_2^{-h}; w_1\overrightarrow{C}v_1^-) \neq \emptyset$. Then by Claim 4.3(i), $|N_G(v_2^{-h}; w_1\overrightarrow{C}v_1^-)| = 1$, and this implies that $G[V(w_1\overrightarrow{C}v_1^-) \cup \{v_2^{-h}\}]$ contains an induced path Q_1 of order at least 3 with an end v_2^{-h}. By Claim 4.10(ii), (4.1) and (4.5), we also see that $Q_2 = v_2^{-h}\overrightarrow{C}v_2x_1$ is an induced path of order at least 3 (see the left of Figure 7). Hence by Lemma 1, $N_G(Q_1 - v_2^{-h}; Q_2 - v_2^{-h}) \neq \emptyset$, which contradicts Fact 4.4, Claim 4.9 or the choice of v_2^{-h}. Thus $N_G(v_2^{-h}; w_1\overrightarrow{C}v_1^-) = \emptyset$, and so $v_2^{-h}w_1^- \in E(G)$.

Consider the paths $Q_1' = w_1^-w_1w_1^+$ and $Q_2' = w_1^-v_2^{-h}v_2^{-(h-1)}$. By Claim 4.9, Q_1' is an induced path. Recall that $N_G(v_2^{-i}; w_1\overrightarrow{C}v_1^-) = \emptyset$ for $i \in \{1, 2\}$. This together with Claim 4.9 leads to $N_G(Q_1' - w_1^-; Q_2' - w_1^-) = \emptyset$ (see also the center of Figure 7). Hence by Lemma 1, we have $v_2^{-(h-1)}w_1^- \in E(G)$, and the choice of v_2^{-h} implies that $h = 1$ and $v_2w_1^- \in E(G)$. Then, again by applying Lemma 1 as $(a, Q_1, Q_2) = (w_1^-, Q_1', w_1^-v_2x_1)$, we have $N_G(\{w_1, w_1^+\}; \{v_2, x_1\}) \neq \emptyset$ or $x_1w_1^- \in E(G)$. Then Fact 4.4 and Claim 4.9 yield that $x_1w_1^- \in E(G)$, which contradicts (4.4). □

Claim 4.12 $z_1w_1 \notin E(G)$.

Proof. Suppose that $z_1w_1 \in E(G)$, and consider the paths $Q_1 = v_2^- z_1 w_1$ (note that by Claim 4.10(i) and (4.5), $z_1 v_2^- \in E(G)$) and $Q_2 = v_2^- v_2 x_1$. By Claim 4.11, Q_1 is an induced path in G. By (4.1), Q_2 is also an induced path. Hence by Lemma 1, $N_G(Q_1 - v_2^-; Q_2 - v_2^-) \neq \emptyset$, which contradicts Fact 4.4, Claim 4.9, Claim 4.10(ii) or (4.1). $\square$

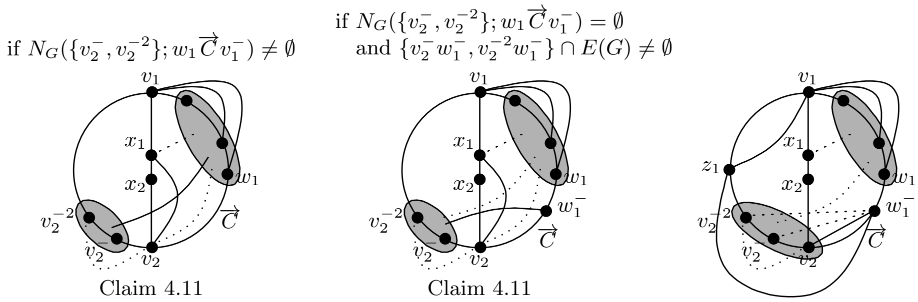


Figure 7: The cycle C in Case 2

Claim 4.13 $\{z_1 w_1^-, v_2 w_1^-\} \subseteq E(G)$.

Proof. We first show that $z_1 w_1^- \in E(G)$. We consider the paths $Q_1 = v_1 w_1 w_1^-$ and $Q_2 = v_1 z_1 v_2^-$ (note that by Claim 4.10(ii) and (4.5), $z_1 v_2^- \in E(G)$). By Claim 4.9, Q_1 is an induced path. By (4.5) and the choice of z_1, Q_2 is also an induced path. Hence by Lemma 1, $N_G(\{z_1, v_2^-\}; \{w_1, w_1^-\}) \neq \emptyset$. Combining this with Claims 4.11 and 4.12, we get $z_1 w_1^- \in E(G)$.

To show that $v_2 w_1^- \in E(G)$, consider the paths $Q_1' = z_1 v_2^- v_2$ and $Q_2' = z_1 w_1^- w_1$. By Claim 4.10(ii) and (4.5), Q_1' is an induced path of order 3. By Claim 4.12, Q_2' is also an induced path. Hence by Lemma 1, $N_G(\{v_2, v_2^-\}; \{w_1, w_1^-\}) \neq \emptyset$. This together with Claims 4.9 and 4.11 implies that $v_2 w_1^- \in E(G)$. $\square$

The graph illustrated in the right of Figure 7 is a current situation. By Claim 4.13, $z_1 w_1^- \in E(G)$. This together with Claim 4.11 implies that $z_1 \notin \{v_2^-, v_2^{-2}\}$. In particular, $|z_1 \overrightarrow{C} v_2| \geq 4$. Now we consider the paths $Q_1 = v_2 w_1^- w_1$ (note that by Claim 4.13, $v_2 w_1^- \in E(G)$) and $Q_2 = v_2 v_2^- v_2^{-2}$. By Claim 4.9, Q_1 is an induced path. By Claim 4.10(ii), Q_2 is also an induced path. Hence by Lemma 1, $N_G(\{w_1^-, w_1\}; \{v_2^-, v_2^{-2}\}) \neq \emptyset$, which contradicts Claim 4.11.

This completes the proof of Theorem 2. $\square$

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